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Large N asymptotics of orthogonal polynomials From integrability to algebraic geometry

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1 Introduction

Random matrices play an important role in physics and mathematics [30, 19, 6, 14, 25, 34, 13]. It has been observed more and more in the recent years how deeply random matrices are related to integrability (τ-functions), and algebraic geometry.

Here, we consider the computation of large n asymptotics for orhogonal polynomials as an example of a problem where the concepts of integrability, isomonodromy and algebraic geometry appear and combine.

The method presented here below, is not, to that date, rigorous mathematicaly. It is based on the asumption that an integral with a large number of variables can be approximated by a saddle-point method. This asumption was never proven rigorously, it is mostly based on "physical intuition". However, the results given by that method have been rigorously proven by another method, namely the Riemann–Hilbert method [7, 8, 11, 12]. The method presented below was presented in many works [17, 16, 2, 20, 18].

2 Definitions

Here we consider the 1-Hermitean matrix model with polynomial potential:

$$\begin{aligned} Z_N &:= \int_{H_N} dM\, \mathbf{e}^{-N \operatorname{tr} V(M)} \\ &= \int_{\mathbb{R}^N} dx_1 \dots dx_N \ \left(\Delta(x_1,\dots,x_N)\right)^2 \prod_{i=1}^{N} \mathbf{e}^{-NV(x_i)} \end{aligned} \tag{2-1}$$

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where $\Delta(x_1, \ldots, x_N) := \prod_{i>j}(x_i - x_j)$, and the x_i's are the eigenvalues of the matrix M, and $V(x)$ is a polynomial called the potential:

$$V(x) = \sum_{k=0}^{\deg V} g_k x^k \tag{2-2}$$

Remark 2.1 *All the calculations which are presented below, can be extended to a more general setting, with no big fundamental changes:*

- one can consider $V'(x)$ any rational fraction [3] instead of polynomial, in particular one can add logarithmic terms to the potential $V(x)$.

- one can consider arbitrary paths (or homology class of paths) of integrations Γ^N insteaf of $\mathbb{R}^N$, in particular finite segments [1] ...

- one can study non hermitean matrix models [20], where the Vandermonde Δ^2 is replaced by Δ^β where $\beta = 1, 2, 4$.

- one can consider multi-matrix models, in particular 2-matrix model [2, 17, 16].

3 Orthogonal polynomials

Consider the family of monic polynomials $p_n(x) = x^n + O(x^{n-1})$, defined by the orthogonality relation:

$$\int_{\mathbb{R}} p_n(x) p_m(x) \mathrm{e}^{-NV(x)} dx = h_n \delta_{nm} \tag{3-3}$$

It is well known that the partition function is given by [30]:

$$Z_N = N! \prod_{n=0}^{N-1} h_n \tag{3-4}$$

Such an orthogonal family always exists if the integration path is $\mathbb{R}$ or a subset of $\mathbb{R}$, and if the potential is a real polynomial. In the more general setting, the orthogonal polynomials "nearly always" exist (for arbitrary potentials, the set of paths for which they don't exist is enumerable).

We define the kernel:

$$K(x, y) := \sum_{n=0}^{N-1} \frac{p_n(x) p_n(y)}{h_n} \tag{3-5}$$

One has the following usefull theorems:

Theorem 3.1 *Dyson's theorem [15]: any correlation function of eigenvalues, can be written in terms of the kernel K:*

$$\rho(\lambda_1, \ldots, \lambda_k) = \det(K(\lambda_i, \lambda_j)) \tag{3-6}$$

Thus, if one knows the orthogonal polynomials, then one knows all the correlation functions.

Theorem 3.2 *Christoffel-Darboux theorem [30, 32]: The kernel $K(x,y)$ can be written:*

$$K(x,y) = \gamma_N \, \frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x-y} \tag{3-7}$$

Thus, if one knows the polynomials p_N and p_{N-1}, then one knows all the correlation functions.

Our goal now, is to find large N "strong" asymptotics for p_N and p_{N-1}, in order to have the large N behaviours of any correlation functions.

Notation: we define the wave functions:

$$\psi_n(x) := \frac{1}{\sqrt{h_n}}\, p_n(x)\, \mathrm{e}^{-\frac{N}{2}V(x)} \tag{3-8}$$

they are orthonormal:

$$\int \psi_n(x)\psi_m(x) = \delta_{nm} \tag{3-9}$$

4 Differential equations and integrability

It can be proven that (ψ_n, ψ_{n-1}) obey a differential equation of the form [9, 7, 30, 33, 5]:

$$-\frac{1}{N}\frac{\partial}{\partial x}\begin{pmatrix}\psi_n(x)\\ \psi_{n-1}(x)\end{pmatrix} = \mathcal{D}_n(x)\begin{pmatrix}\psi_n(x)\\ \psi_{n-1}(x)\end{pmatrix} \tag{4-10}$$

where $\mathcal{D}_n(x)$ is a 2×2 matrix, whose coefficients are polynomial in x, of degree at most $\deg V'$. (In case V' is a rational function, then $\mathcal{D}$ is a rational function with the same poles).

(ψ_n, ψ_{n-1}) also obeys differential equations with respect to the parameters of the model [7, 5], i.e. the coupling constants, i.e. the g_k's defined in 2-2:

$$\frac{1}{N}\frac{\partial}{\partial g_k}\begin{pmatrix}\psi_n(x)\\ \psi_{n-1}(x)\end{pmatrix} = \mathcal{U}_{n,k}(x)\begin{pmatrix}\psi_n(x)\\ \psi_{n-1}(x)\end{pmatrix} \tag{4-11}$$

where $\mathcal{U}_{n,k}(x)$ is a 2×2 matrix, whose coefficients are polynomial in x, of degree at most k.

It is also possible to find some discrete recursion relation in n (see [5]).

The compatibility of these differential systems, i.e. $\frac{\partial}{\partial x}\frac{\partial}{\partial g_k} = \frac{\partial}{\partial g_k}\frac{\partial}{\partial x}$, $\frac{\partial}{\partial g_j}\frac{\partial}{\partial g_k} = \frac{\partial}{\partial g_k}\frac{\partial}{\partial g_j}$, as well as compatibility with the discrete recursion, imply **integrability**, and allows to define a τ-function [27, 5].

We define the spectral curve as the locus of eigenvalues of $\mathcal{D}_n(x)$:

$$E_n(x,y) := \det(y\mathbf{1} - \mathcal{D}_n(x)) \tag{4-12}$$

Remark 4.1 In the 1-hermitean-matrix model, $\mathcal{D}_n$ is a 2×2 matrix, and thus $\deg_y E_n(x,y) = 2$, i.e. the curve $E_n(x,y) = 0$ is an **hyperelliptical curve**. In other matrix models, one gets algebraic curves which are not hyperelliptical.

Remark 4.2 What we will se below, is that the curve $E_N(x,y)$ has a large N limit $E(x,y)$, which is also an hyperelliptical curve. In general, the matrix $\mathcal{D}_N(x)$ has no large N limit.

5 Riemann-Hilbert problems and isomonodromies

The 2×2 system $\mathcal{D}_N$ has 2 independent solutions:

$$-\frac{1}{N}\frac{\partial}{\partial x}\begin{pmatrix} \psi_n(x) \\ \psi_{n-1}(x) \end{pmatrix} = \mathcal{D}_n(x)\begin{pmatrix} \psi_n(x) \\ \psi_{n-1}(x) \end{pmatrix} \quad , \quad -\frac{1}{N}\frac{\partial}{\partial x}\begin{pmatrix} \phi_n(x) \\ \phi_{n-1}(x) \end{pmatrix} = \mathcal{D}_n(x)\begin{pmatrix} \phi_n(x) \\ \phi_{n-1}(x) \end{pmatrix} \tag{5-13}$$

where the wronskian is non-vanishing: $\det\begin{pmatrix} \psi_n(x) & \phi_n(x) \\ \psi_{n-1}(x) & \phi_{n-1}(x) \end{pmatrix} \neq 0$.

We define the matrix of fundamental solutions:

$$\Psi_n(x) := \begin{pmatrix} \psi_n(x) & \phi_n(x) \\ \psi_{n-1}(x) & \phi_{n-1}(x) \end{pmatrix} \tag{5-14}$$

it obeys the same differential equation:

$$-\frac{1}{N}\frac{\partial}{\partial x}\Psi_n(x) = \mathcal{D}_n(x)\,\Psi_n(x) \tag{5-15}$$

Here, the second solution can be constructed explicitely:

$$\phi_n(x) = \mathbf{e}^{+\frac{N}{2}V(x)}\int \frac{dx'}{x-x'}\,\psi_n(x')\mathbf{e}^{-\frac{N}{2}V(x')} \tag{5-16}$$

Notice that $\phi_n(x)$ is discontinuous along the integration path of x' (i.e. the real axis in the most simple case), the discontinuity is simply $2i\pi\psi_n(x)$. In terms of fundamental solutions, one has the jump relation:

$$\Psi_n(x+i0) = \Psi_n(x-i0)\begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix} \tag{5-17}$$

Finding an invertible piecewise analytical matrix, with given large x behaviours, with given jumps on the borders between analytical domains, is called a **Riemann–Hilbert problem** [7, 8, 4].

It is known that the Riemann–Hilbert problem has a unique solution, and that if two R-H problems differ by ϵ (i.e. the difference between jumps and behaviours at ∞ is bounded by ϵ), then the two solutions differ by at most ϵ (roughly speeking, harmonic functions have their extremum on the boundaries). Thus, this approach can be used [7, 11, 12] in order to find large N asymptotics of orthogonal polynomials: The authors of [7] considered a guess for the asymptotics, which satisfies another R-H problem, which differs from this one by $O(1/N)$.

Notice that the jump matrix in 5-17 is independent of x, of n and of the potential, it is a constant. The jump matrix is also called a monodromy, and the fact that the monodromy is a constant, is called **isomonodromy** property [27].

Consider an invertible, piecewise analytical matrix $\Psi_n(x)$, with appropriate behaviours[2] at ∞, which satisfies 5-17, then, it is clear that the matrix $-\frac{1}{N}\Psi'_n(x)(\Psi_n(x))^{-1}$, has no discontinuity, and given its behaviour at ∞, it must be a polynomial. Thus, we can prove that $\Psi_n(x)$ must satisfy a differential system $\mathcal{D}_n(x)$ with polynomial coefficients. Similarly, the fact that the monodromy is independent of g_k and n implies the deformation equations, as well as the discrete recursion relations.

Thus, the isomonodromy property, implies the existence of compatible differential systems, and integrability [6, 24, 26, 27, 33, 5].

6 WKB–like asymptotics and spectral curve

Let us look for a formal solution of the form:

$$\Psi_N(x) = A_N(x)\,\mathrm{e}^{-NT(x)}B_N \tag{6-18}$$

where $T(x) = \mathrm{diag}(T_1(x), T_2(x))$ is a diagonal matrix, and B_N is independent of x. The differential system $\mathcal{D}_N(x)$ is such that:

$$\begin{aligned}\mathcal{D}_N(x) &= -\frac{1}{N}\Psi'_N\Psi_N^{-1} = A_N(x)T'(x)A_N^{-1}(x) - \frac{1}{N}A'_N(x)A_N^{-1}(x)\\ &= A_N(x)T'(x)A_N^{-1}(x) + O(\frac{1}{N})\end{aligned} \tag{6-19}$$

this means, that, under the asumption that $A_N(x)$ has a large N limit $A(x)$, $T'_1(x)$ and $T'_2(x)$ are the large N limits of the eigenvalues of $\mathcal{D}_N(x)$.

With such an hypothesis, one gets for the orthogonal polynomials:

$$\psi_N(x) \sim A_{11}\mathrm{e}^{-NT_1(x)}B_{1,1} + A_{12}\mathrm{e}^{-NT_2(x)}B_{2,1} \tag{6-20}$$

We are now going to show how to derive such a formula.

7 Orthogonal polynomials as matrix integrals

7.1 Heine's formula

Theorem 7.1 *Heine's theorem [32]. The orthogonal polynomials $p-n(x)$ are given by:*

$$p_n(\xi) = \frac{\int dx_1\dots dx_N\ \prod_{i=1}^N(\xi - x_i)\ (\Delta(x_1,\dots,x_N))^2\ \prod_{i=1}^N \mathrm{e}^{-NV(x_i)}}{\int dx_1\dots dx_N\ (\Delta(x_1,\dots,x_N))^2\ \prod_{i=1}^N \mathrm{e}^{-NV(x_i)}}$$

[2] The behaviours at ∞ are far beyond the scope of this short lecture. They are easily obtained by computing $\phi_n(x)$ by saddle point method at large x.

$$= \langle \det(\xi \mathbf{1} - M) \rangle \tag{7-21}$$

i.e. the orthogonal polynomial is the average of the characteristic polynomial of the random matrix.

Thus, we can define the orthogonal polynomials as matrix integrals, similar to the partition function Z define in 2-1.

7.2 Another matrix model

Define the potential:

$$V_h(x) := V(x) - h \ln(\xi - x) \tag{7-22}$$

and the partition function:

$$Z_n(h,T) := \mathbf{e}^{-\frac{n^2}{T^2} F_n(h,T)} := \int dx_1 \dots dx_n \, (\Delta(x_1,\dots,x_n))^2 \prod_{i=1}^n \mathbf{e}^{-\frac{n}{T} V_h(x_i)} \tag{7-23}$$

i.e. $Z_N(0,1) = Z$ is our initial partition function.

Heine's formula reads:

$$p_n(\xi) = \frac{Z_n(\frac{1}{N}, \frac{n}{N})}{Z_n(0, \frac{n}{N})} = \mathbf{e}^{-N^2 (F_n(\frac{1}{N}, \frac{n}{N}) - F_n(0, \frac{n}{N}))} \tag{7-24}$$

The idea, is to perform a Taylor expansion in h close to 0 and T close to 1.

7.2.1 Taylor expansion

We are interested in $n = N$ and $n = N-1$, thus $T = \frac{n}{N} = 1 + \frac{n-N}{N} = 1 + O(1/N)$ and $h = 0$ or $h = 1/N$, i.e. $h = O(1/N)$:

$$T = 1 + O(1/N) \quad , \quad h = O(1/N) \tag{7-25}$$

Roughly speaking:

$$\begin{aligned} p_n(\xi) &\sim \mathbf{e}^{-N^2 \left(h \frac{\partial F}{\partial h} + (T-1) h \frac{\partial^2 F}{\partial h \partial T} + \frac{h^2}{2} \frac{\partial^2 F}{\partial h^2} + O(1/N^3) \right)} \\ &\sim \mathbf{e}^{-N \frac{\partial F}{\partial h}} \, \mathbf{e}^{-(n-N) \frac{\partial^2 F}{\partial h \partial T}} \, \mathbf{e}^{-\frac{1}{2} \frac{\partial^2 F}{\partial h^2}} \, (1 + O(1/N)) \end{aligned} \tag{7-26}$$

where all the derivatives are computed at $T = 1$ and $h = 0$.

7.2.2 Topological expansion

Imagine that F_n has a $1/n^2$ expansion of the form:

$$F = F^{(0)} + \frac{1}{n^2} F^{(1)} + O(\frac{1}{n^3}) \tag{7-27}$$

where all $F^{(0)}$ and $F^{(1)}$ are analytical functions of T and h, than one needs only $F^{(0)}$ in order to compute the asymptotics 7-26.

Actualy, that hypothesis is not always true. It is wrong in the so called "mutlicut" case. But it can be adapted in that case, we will come back to it in section 11.2. For the moment, let us conduct the calculation only with $F^{(0)}$.

8 Computation of derivatives of $F^{(0)}$

We have defined:

$$Z_n(h,T) = \mathbf{e}^{-\frac{n^2}{T^2}F_n(h,T)} = \int dM_{n\times n}\mathbf{e}^{-\frac{n}{T}\mathrm{tr}\,V(M)}\,\mathbf{e}^{h\frac{n}{T}\ln(\xi-M)} \tag{8-28}$$

this implies that:

$$-\frac{n^2}{T^2}\frac{\partial F_n}{\partial h} = \left\langle\frac{n}{T}\mathrm{tr}\,\ln(\xi-M)\right\rangle_{V_h} \tag{8-29}$$

i.e.

$$\frac{\partial F_n}{\partial h} \underset{(8-30)}{=} -\frac{T}{n}\left\langle\mathrm{tr}\,\ln(\xi-M)\right\rangle_{V_h}$$

It is a primitive of $-\frac{T}{n}\left\langle\mathrm{tr}\,\ln(x-M)\right\rangle_{V_h}$, which behaves as $-\frac{T}{n}\ln x + O(1/x)$ at large x. Therefore, we define the resolvent $W(x)$:

$$W(x) := \frac{T}{n}\left\langle\mathrm{tr}\,\frac{1}{x-M}\right\rangle_{V_h} \tag{8-31}$$

Notice that it depends on ξ through the potential V_h, i.e. through the average $< . >$. And we define the effective potential:

$$V_{\text{eff}}(x) = V_h(x) - 2T\ln x - 2\int_\infty^x (W(x') - \frac{T}{x'})dx' \tag{8-32}$$

which is a primitive of $V_h'(x) - 2W(x)$. Thus , we have:

$$\frac{\partial F_n}{\partial h} = \frac{1}{2}\left(V_{\text{eff}}(\xi) - V_h(\xi)\right) \tag{8-33}$$

We also introduce:

$$\Omega(x) := \frac{\partial W(x)}{\partial T} \quad , \quad \ln\Lambda(x) := \ln x + \int_\infty^x (\Omega(x') - \frac{1}{x'})dx' = -\frac{1}{2}\frac{\partial}{\partial T}V_{\text{eff}}(x) \tag{8-34}$$

$$H(x,\xi) := \frac{\partial W(x)}{\partial h} \quad , \quad \ln H(\xi) := \int_\infty^\xi H(x',\xi)dx' \tag{8-35}$$

i.e.

$$\frac{\partial^2 F_n}{\partial h^2} = -\ln H(\xi) \quad , \quad \frac{\partial^2 F_n}{\partial h \partial T} = -\ln \Lambda(\xi) \tag{8-36}$$

With these notations, the asymptotics are:

$$\psi_n(\xi) \sim \sqrt{H(\xi)}\ \left(\Lambda(\xi)\right)^{n-N}\ \mathbf{e}^{-\frac{N}{2}V_{\mathrm{eff}}(\xi)}\ (1+O(1/N)) \tag{8-37}$$

Now, we are going to compute W, Λ, H, etc, in terms of geometric properties of an hyperelliptical curve.

Remark 8.1 This is so far only a sketch of the derivation, valid only in the 1-cut case. In general, F_n has no $1/n^2$ expansion, and that case will be addressed in section 11.2.

Remark 8.2 These asymptoics are of the form of 6-18 in section.6, and thus, $\frac{1}{2}V'(x) - W(x)$ is the limit of the eigenvalues of $\mathcal{D}_N(x)$.

9 Saddle point method

There exists many ways of computing the resolvent and its derivatives with respect to h, T, or other parameters. The loop equation method is a very good method, but there is not enough time to present it here. There are several saddle-point methods, which all coincide to leading order. We are going to present one of them, very intuitive, but not very rigorous on a mathematical ground, and not very appropriate for next to leading computations. However, it gives the correct answer to leading order.

Write:

$$Z_n(h,T) = \mathbf{e}^{-\frac{n^2}{T^2}F_n(h,T)} = \int dx_1 \dots dx_n \mathbf{e}^{-\frac{n^2}{T^2}\mathcal{S}(x_1,\dots,x_n)} \tag{9-38}$$

where the action is:

$$\mathcal{S}(x_1,\dots,x_n) := \frac{T}{n}\sum_{i=1}^{n} V_h(x_i) - 2\frac{T^2}{n^2}\sum_{i>j}\ln(x_i - x_j) \tag{9-39}$$

The saddle point method consists in finding configurations $x_i = \overline{x}_i$ where $\mathcal{S}$ is extremal, i.e.

$$\forall i = 1,\dots n, \qquad \left.\frac{\partial \mathcal{S}}{\partial x_i}\right|_{x_j=\overline{x}_j} = 0 \tag{9-40}$$

i.e., we have the **saddle point equation**:

$$\forall i = 1,\dots n, \qquad V_h'(\overline{x}_i) = 2\frac{T}{n}\sum_{j\neq i}\frac{1}{\overline{x}_i - \overline{x}_j} \tag{9-41}$$

The saddle point approximation[3] consists in writting:

$$Z_n(h,T) \sim \frac{1}{\sqrt{\det\left(\frac{\partial \mathcal{S}}{\partial x_i \partial x_j}\right)}}\ \mathbf{e}^{-\frac{n^2}{T^2}\mathcal{S}(\overline{x}_1,\dots,\overline{x}_n)}\ (1+O(1/n)) \tag{9-42}$$

where $(\overline{x}_1,\dots,\overline{x}_n)$ is the solution of the saddlepoint equation which minimizes $\Re\mathcal{S}$.

Remark 9.1 The saddle point equation may have more than one minimal solution $(\overline{x})$.
- in particular if $\xi \in \mathbb{R}$, there are two solutions, complex conjugate of each other.
- in the multicut case, there are many saddlepoints with near-minimal action.

In all cases, one needs to sum over all the saddle points. Let us call $\{\overline{x}\}_I$, the collection of saddle points. We have:

$$Z_n \sim \sum_I \frac{C_I}{\sqrt{\mathcal{S}''(\{\overline{x}\}_I)}}\ \mathbf{e}^{-\frac{n^2}{T^2}\mathcal{S}(\{\overline{x}\}_I)}\ (1+O(1/n)) \tag{9-43}$$

Each saddle point $\{\overline{x}\}_I$ corresponds to a particular minimal n-dimensional integration path in $\mathbb{C}^n$,noted Γ_I, and the coefficients $C_I \in \mathbb{Z}$ are such that:

$$\mathbb{R}^n = \sum_I C_I \Gamma_I \tag{9-44}$$

10 Solution of the saddlepoint equation

We recall the saddle point equation:

$$\forall i = 1,\dots n, \qquad V_h'(\overline{x}_i) = 2\frac{T}{n}\sum_{j\neq i}\frac{1}{\overline{x}_i - \overline{x}_j} \tag{10-45}$$

We introduce the function:

$$\omega(x) := \frac{T}{n}\sum_{j=1}^{n}\frac{1}{x-\overline{x}_j} \tag{10-46}$$

in the large N limit, $\omega(x)$ is expected to tend toward the resolvent, at least in the case there is only one minimal saddle point. Indeed, the $\overline{x}_i$'s are the position of the eigenvalues minimizing the action, i.e. the most probable positions of eigenvalues of M, and thus 10-46 should be close to $\frac{T}{n}\mathrm{tr}\,\frac{1}{x-M}$.

10.1 Algebraic method

Compute $\omega^2(x) + \frac{T}{n}\omega'(x)$, you find:

$$\omega^2(x) + \frac{T}{n}\omega'(x) \qquad = \qquad \frac{T^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{(x-\overline{x}_i)(x-\overline{x}_j)} - \frac{T^2}{n^2}\sum_{i=1}^{n}\frac{1}{(x-\overline{x}_i)^2}$$

[3]The validity of the saddle point approximation is not proven rigorously for large number of variables. But here, we have many evidences that we can trust the results it gives. The asymptotics we are going to find have been proven rigorously by other methods. Basicaly, it is expected to work because the number of variables n is small compared to the large parameter n^2 in the action.

$$
\begin{aligned}
&= \frac{T^2}{n^2}\sum_{i\neq j}^{n}\frac{1}{(x-\overline{x}_i)(x-\overline{x}_j)} \\
&= \frac{T^2}{n^2}\sum_{i\neq j}^{n}\left(\frac{1}{x-\overline{x}_i}-\frac{1}{x-\overline{x}_j}\right)\frac{1}{\overline{x}_i-\overline{x}_j} \\
&= \frac{2T^2}{n^2}\sum_{i=1}^{n}\frac{1}{x-\overline{x}_i}\sum_{j\neq i}^{n}\frac{1}{\overline{x}_i-\overline{x}_j} \\
&= \frac{T}{n}\sum_{i=1}^{n}\frac{V_h'(\overline{x}_i)}{x-\overline{x}_i} \\
&= \frac{T}{n}\sum_{i=1}^{n}\frac{V_h'(x)-(V_h'(x)-V_h'(\overline{x}_i))}{x-\overline{x}_i} \\
&= V_h'(x)\omega(x)-\frac{T}{n}\sum_{i=1}^{n}\frac{V_h'(x)-V_h'(\overline{x}_i)}{x-\overline{x}_i} \\
&= (V'(x)-\frac{h}{x-\xi})\omega(x)-\frac{T}{n}\sum_{i=1}^{n}\frac{V'(x)-V'(\overline{x}_i)}{x-\overline{x}_i}+h\frac{\omega(\xi)}{x-\xi}
\end{aligned}
\quad (10-47)
$$

i.e. we get the equation:

$$\omega^2(x)+\frac{T}{n}\omega'(x)=V'(x)\omega(x)-P(x)-h\frac{\omega(x)-\omega(\xi)}{x-\xi} \quad (10\text{-}48)$$

where $P(x):=\frac{T}{n}\sum_{i=1}^{n}\frac{V'(x)-V'(\overline{x}_i)}{x-\overline{x}_i}$ is a polynomial in x of degree at most $\deg V-2$.

In the large N limit, if we assume[4] that we can drop the $1/NW'(x)$ term, we get an algebraic equation, which is in this case an hyperelliptical curve. In particular at $h=0$ and $T=1$:

$$\omega(x)=\frac{1}{2}\left(V'(x)-\sqrt{V'^2(x)-4P(x)}\right) \quad (10\text{-}49)$$

The properties of this algebraic equation have been studied by many authors, and the T and h derivatives, as well as other derivatives were computed in various works. Here, we briefly sketch the method. See [29, 28, 21] for more details.

10.2 Linear saddle point equation

In the large N limit, both the average density of eigenvalues, and the density of $\overline{x}$ tend towards a continuous compact support density $\overline{\rho}(x)$. In that limit, the resolvent is given by:

$$\omega(x)=T\int_{\mathrm{supp}\,\overline{\rho}}\frac{\overline{\rho}(x')\,dx'}{x-x'} \quad (10\text{-}50)$$

[4]It is possible to do the calculation without droping the $1/N$ term. One gets a Ricati equation, which is equivalent to a Schroedinger equation. If one is interested in a large N limit for the resolvent, the asymptotic analysis of that Schroedinger equation (Stokes phenomenon) gives, to leading order, the same thing as when one drops the $1/N$ term. If one whishes to go beyond leading order, many subtleties occur.

i.e.

$$\forall x \in \text{supp } \overline{\rho}, \qquad \overline{\rho}(x) = -\frac{1}{2i\pi T}(\omega(x+i0) - \omega(x-i0)) \tag{10-51}$$

and the saddle point equation 10-45, becomes a linear functional equation:

$$\forall x \in \text{supp } \overline{\rho}, \qquad V_h'(x) = \omega(x+i0) + \omega(x-i0) \tag{10-52}$$

The advantage of that equation, is that it is linear in ω, and thus in $\overline{\rho}$. The nonlinearity is hidden in supp $\overline{\rho}$.

10.2.1 Example: One cut

If the support of $\overline{\rho}$ is a single interval:

$$\text{supp } \overline{\rho} = [a,b] \quad , \quad a < b \tag{10-53}$$

then, look for a solution of the form:

$$\omega(x) = \frac{1}{2}\left(V_h'(x) - M_h(x)\sqrt{(x-a)(x-b)}\right) \tag{10-54}$$

The saddle point equation 10-52 implies that $M_h(x+i0) = M_h(x-i0)$, i.e. M_h has no discontinuities, and because of its large x behaviour, as well as its behaviours near ξ, it must be a rational function of x, with a simple pole at $x = \xi$. M_h, a and b are entirely determined by their behaviours near poles, i.e.:

$$\omega(x) \underset{x\to\infty}{\sim} \frac{T}{x} \tag{10-55}$$

$$\omega(x) \underset{x\to\xi}{\sim} \text{regular} \quad \longrightarrow M_h(x) \underset{x\to\xi}{\sim} -\frac{h}{x-\xi} \tag{10-56}$$

Thus, one may write:

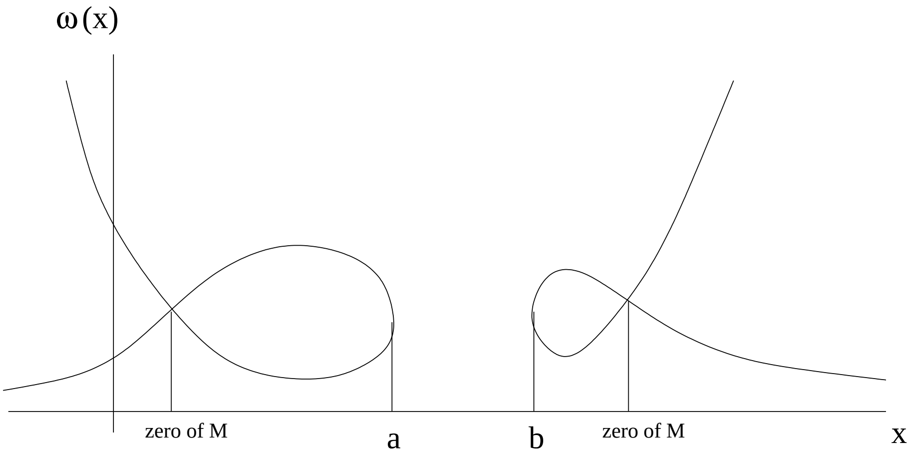
$$\omega(x) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x-a)(x-b)} - \frac{h}{x-\xi}\left(1 - \frac{\sqrt{(x-a)(x-b)}}{\sqrt{(\xi-a)(\xi-b)}}\right)\right) \tag{10-57}$$

where $M(x)$ is now a polynomial (which still depends on h and T and the other parameters), it is such that:

$$M(x) = \underset{x\to\infty}{\text{Pol}} \, \frac{V'(x)}{\sqrt{(x-a)(x-b)}} \tag{10-58}$$

The density is thus:

$$\overline{\rho}(x) = \frac{1}{2\pi T} M_h(x)\sqrt{(x-a)(b-x)} \quad , \quad \text{supp } \overline{\rho} = [a,b] \tag{10-59}$$



10.2.2 Multi-cut solution

Let us assume that the support of $\overline{\rho}$ is made of s separated intervals:

$$\text{supp } \overline{\rho} = \cup_{i=1}^{s}[a_i, b_i] \tag{10-60}$$

then, for any sequence of integers $n_1, n_2, \ldots, n_s$ such that $\sum_{i_1}^{s} n_i = n$, it is possible to find a solution for the saddle point equation. That solution obeys 10-52, as well as the conditions:

$$\forall i = 1, \ldots, s \quad , \quad \int_{a_i}^{b_i} \rho(x) dx = T\frac{n_i}{N} \tag{10-61}$$

The solution of the saddle point equation can be described as follows:
let the polynomial $\sigma(x)$ be defined as:

$$\sigma(x) := \prod_{i=1}^{s}(x - a_i)(x - b_i) \tag{10-62}$$

The solution of the saddle point equation 10-52, is of the form:

$$\omega(x) = \frac{1}{2}\left(V_h'(x) - M_h(x)\sqrt{\sigma(x)}\right) \tag{10-63}$$

where $M_h(x)$ is a rational function of x, with a simple pole at $x = \xi$. M_h, and $\sigma(x)$ are entirely determined by their behaviours near poles, i.e.:

$$\omega(x) \underset{x\to\infty}{\sim} \frac{T}{x} \tag{10-64}$$

$$\omega(x) \underset{x\to\xi}{\sim} \text{regular} \quad \longrightarrow M_h(x) \underset{x\to\xi}{\sim} -\frac{h}{x - \xi} \tag{10-65}$$

and by the conditions that:

$$\forall i = 1, \ldots, s \quad , \quad \int_{a_i}^{b_i} M_h(x)\sqrt{\sigma(x)} dx = 2i\pi T\frac{n_i}{n} \tag{10-66}$$

10.3 Algebraic geometry: hyperelliptical curves

Consider the curve given by:

$$\omega(x) = \frac{1}{2}\left(V_h'(x) - M_h(x)\sqrt{(x-a)(x-b)}\right) \tag{10-67}$$

It has two sheets, i.e. for each x, there are two values of $\omega(x)$, depending on the choice of sign of the square-root.

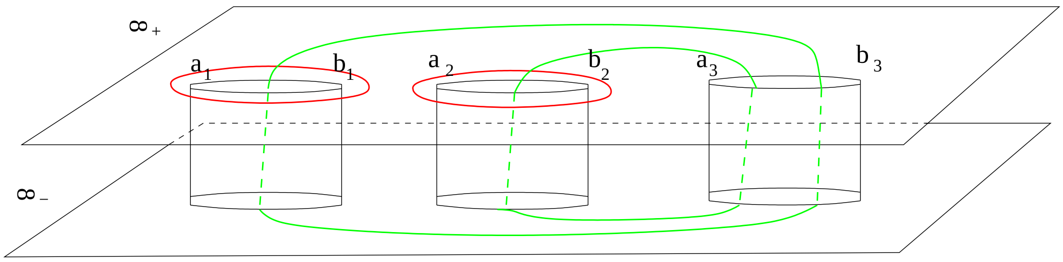
- In the physical sheet (choice $+\surd$), it behaves near ∞ like $\omega(x) \sim T/x$

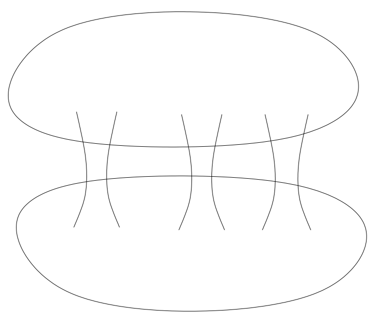
- In the second sheet (choice $-\surd$), it behaves near ∞ like $\omega(x) \sim V_h'(x)$

Since $\omega(x)$ is a complex valued, analytical function of a cmplex variable x, the curve can be thought of as the embedding of a Riemann surface into $\mathbb{C}\times\mathbb{C}$.

I.e. we have a Riemann surface $\mathcal{E}$, with two (monovalued) functions defined on it: $p \in \mathcal{E}\,, \to x(p) \in \mathbb{C}$, and $p \in \mathcal{E}\,, \to \omega(p) \in \mathbb{C}$. For each x, there are two $p \in \mathcal{E}$ such that $x(p) = x$, and this is why there are two values of $\omega(x)$.

Each of the two sheets is homeomorphic to the complex plane, cut along the segments $[a_i, b_i]$, and the two sheets are glued together along the cuts. The complex plane, plus its point at infinity, is the Riemann sphere. Thus, our curve $\mathcal{E}$, is obtained by taking two Riemann spheres, glued together along s circles. It is a genus $s-1$ surface.





10.4 Genus zero case (one cut)

If the curve as genus zero, it is homeomorphic to the Riemann sphere $\mathcal{E} = \mathbb{C}$. One can always choose a rational parametrization:

$$x(p) = \frac{a+b}{2} + \gamma(p + 1/p) \quad , \quad \gamma = \frac{b-a}{4} \tag{10-68}$$

$$\sqrt{(x-a)(x-b)} = \gamma(p - 1/p) \tag{10-69}$$

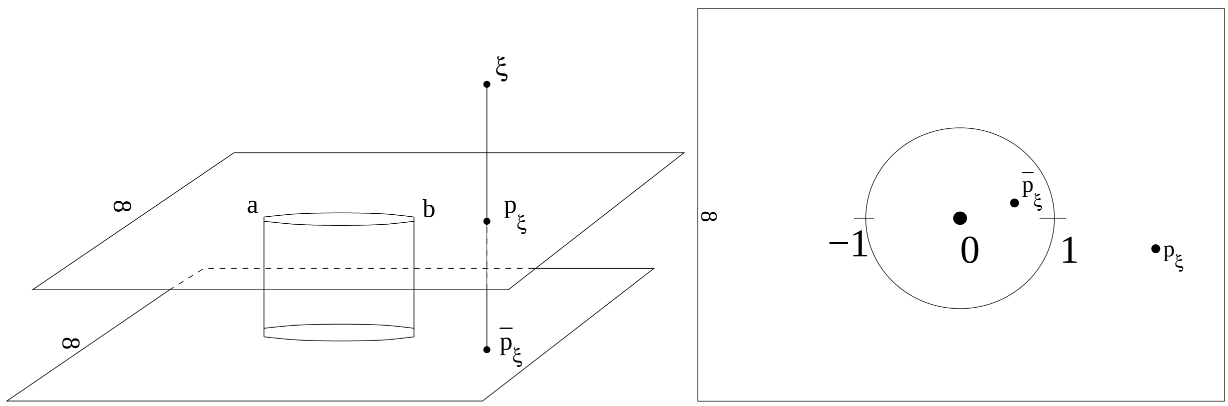
so that ω is a rational function of p.

That representation maps the physical sheet onto the exterior of the unit circle, and the second sheet onto the interior of the unit circle. The unit circle is the image of the two sides of the cut $[a, b]$, and the branchpoints $[a, b]$ are maped to -1 and $+1$. Changing the sign of the square root is equivalent to changing $p \to 1/p$.

The branch points are of course the solutions of $dx/dp = 0$, i.e. $dx(p) = 0$:

$$dx(p) = \gamma \left(1 - \frac{1}{p^2}\right) dp \quad , \quad dx(p) = 0 \leftrightarrow p = \pm 1 \leftrightarrow x(p) = a, b \tag{10-70}$$

There are two points at ∞, $p = \infty$ in the physical sheet, and $p = 0$ in the second sheet.



Since the resolvent $\omega(p)$ is a rational function of p, it is then entirely determined by its behaviour near its poles. the poles are at $p = \infty$, $p = 0$, $p = p_\xi$ and $p = \overline{p}_\xi$ (the two points of $\mathcal{E}$ such that $x(p) = \xi$, such that p_ξ is in the physical sheet, and $\overline{p}_\xi$ is in the second sheet): The boundary conditions:

$$\left\{ \begin{array}{l} \omega(p) \underset{p\to\infty}{\sim} \dfrac{T}{x(p)} \\ \omega(p) \underset{p\to 0}{\sim} V'(x(p)) - \dfrac{T}{x(p)} - \dfrac{h}{x(p)} \\ \omega(p) \underset{p\to\overline{p}_\xi}{\sim} -\dfrac{h}{x(p)-\xi} \\ \omega(p) \underset{p\to p_\xi}{\sim} \text{regular} \end{array} \right. \tag{10-71}$$

10.4.1 T derivative

Now, let us compute $\partial\omega(p)/\partial T$ at $x(p)$ fixed. Eq. 10-71 becomes:

$$\left\{\begin{array}{l} \frac{\partial\omega(p)}{\partial T} \mathop{\sim}_{p\to\infty} \frac{1}{x(p)} \\ \frac{\partial\omega(p)}{\partial T} \mathop{\sim}_{p\to 0} -\frac{1}{x(p)} \\ \frac{\partial\omega(p)}{\partial T} \mathop{\sim}_{p\to\overline{p}_\xi} \text{regular} \\ \frac{\partial\omega(p)}{\partial T} \mathop{\sim}_{p\to p_\xi} \text{regular} \end{array}\right. \tag{10-72}$$

Moreover, we know that $\omega(x)$ has a square-root behaviour near a and b, in $\sqrt{(x-a)(x-b)}$, and a and b depend on T, thus $\partial\omega/\partial T$ may behave in $((x-a)(x-b))^{-1/2}$ near a and b, i.e. $\partial\omega/\partial T$ may have simple poles at $p=\pm 1$.

Finaly, $\partial\omega(p)/\partial T$, has simple poles at $p=1$ and $p=-1$, and vanishes at $p=0$ and $p=\infty$, the only possibility is:

$$\left.\frac{\partial\omega(p)}{\partial T}\right|_{x(p)} = \frac{p}{\gamma(p^2-1)} = \frac{1}{p}\frac{dp}{dx} \tag{10-73}$$

which is better written in terms of differential forms:

$$\left.\frac{\partial\omega(p)}{\partial T}\right|_{x(p)} dx(p) = \frac{dp}{p} = d\ln p \tag{10-74}$$

the RHS is independent of the potential, it is universal.

With the notation 8-34, we have:

$$\Omega(p)dx(p) = \frac{dp}{p} \quad , \quad \Lambda(p) = \gamma p \tag{10-75}$$

10.4.2 h derivative

The h derivative is computed in a very similar way.

$$\left\{\begin{array}{l} \frac{\partial\omega(p)}{\partial h} \mathop{\sim}_{p\to\infty} O(p^{-2}) \\ \frac{\partial\omega(p)}{\partial h} \mathop{\sim}_{p\to 0} -\frac{1}{x(p)} \\ \frac{\partial\omega(p)}{\partial h} \mathop{\sim}_{p\to\overline{p}_\xi} -\frac{1}{x(p)-\xi} \\ \frac{\partial\omega(p)}{\partial h} \mathop{\sim}_{p\to p_\xi} \text{regular} \end{array}\right. \tag{10-76}$$

implies that $\partial\omega/\partial h$ can have poles at $p=\pm 1$ and at $p=\overline{p}_\xi$, and vanishes at $p=0$. The only possibility is:

$$\left.\frac{\partial \omega(p)}{\partial h}\right|_{x(p)} = \frac{-p\,\overline{p}_\xi}{\gamma(p-\overline{p}_\xi)(p^2-1)} \tag{10-77}$$

i.e.

$$\left.\frac{\partial \omega(p)}{\partial h}\right|_{x(p)} dx(p) = \frac{dp}{p} - \frac{dp}{p-\overline{p}_\xi} = d\ln\frac{p}{p-\overline{p}_\xi} \tag{10-78}$$

which again is universal.

With the notation 8-35, we have:

$$H(p,p_\xi)dx(p) = \frac{dp}{p} - \frac{dp}{p-\frac{1}{p_\xi}} \quad , \quad H(p_\xi) = \ln\left(\frac{p_\xi}{p_\xi-\overline{p}_\xi}\right) = -\ln\left(\frac{1}{\gamma}\frac{dx}{dp}(\xi)\right) \tag{10-79}$$

10.5 Higher genus

For general genus, the curve can be parametrized by θ-functions. Like rational functions for genus 0, θ-functions are the building blocks of functions defined on a compact Riemann surface, and any such function is entirely determined by its behaviour near its poles, as well as by its integrals around irreducible cycles. All the previous paragraph can be extended to that case.

Let ∞_+ and ∞_- be the points at infinity, i.e. the two poles of $x(p)$, with ∞_+ in the physical sheet and ∞_- in the second sheet. Let $p=p_\xi$ and $p=\overline{p}_\xi$ be the two points of $\mathcal{E}$ such that $x(p)=\xi$, and with p_ξ in the physical sheet, and $\overline{p}_\xi$ in the second sheet.

The differential form $\omega(p)dx(p)$ is entirely determined by:

$$\left\{\begin{array}{ll}
\omega(p)dx(p) \mathop{\sim}_{p\to\infty_+} T\,\dfrac{dx(p)}{x(p)} & , \quad \mathop{\rm Res}_{\infty_+} \omega(p)dx(p) = -T \\
\omega(p)dx(p) \mathop{\sim}_{p\to\infty_-} dV(x(p)) - T\dfrac{dx(p)}{x(p)} - h\dfrac{dx(p)}{x(p)} & , \quad \mathop{\rm Res}_{\infty_-} \omega(p)dx(p) = T+h \\
\omega(p)dx(p) \mathop{\sim}_{p\to\overline{p}_\xi} -h\dfrac{dx(p)}{x(p)-\xi} & , \quad \mathop{\rm Res}_{\overline{p}_\xi} \omega(p)dx(p) = -h \\
\omega(p)dx(p) \mathop{\sim}_{p\to p_\xi} \text{regular} & , \quad \mathop{\rm Res}_{p_\xi} \omega(p)dx(p) = 0 \\
\displaystyle\oint_{\mathcal{A}_i} \omega(p)dx(p) = T\frac{n_i}{n} = \frac{n_i}{N} &
\end{array}\right. \tag{10-80}$$

Since $\partial\omega/\partial T, h$ can diverge at most like $(x-a_i)^{-1/2}$ near a branch point a_i, and $dx(p)$ has a zero at a_i, the differential form $\partial\omega dx/\partial T, h$ has no pole at the branch points.

10.6 Introduction to algebraic geometry

We introduce some basic concepts of algebraic geometry. We refer the reader to [22, 23] for instance.

Theorem 10.1 *Given two points q_1 and q_2 on the Riemann surface $\mathcal{E}$, there exists a unique differential form $dS_{q_1,q_2}(p)$, with only two simple poles, one at $p = q_1$ with residue $+1$ and one at $p = q_2$ with residue -1, and which is normalized on the $\mathcal{A}_i$ cycles, i.e.*

$$\left\{ \begin{array}{l} \operatorname*{Res}_{p\to q_1} dS_{q_1,q_2}(p) = +1 \\ \operatorname*{Res}_{p\to q_2} dS_{q_1,q_2}(p) = -1 \\ \oint_{\mathcal{A}_i} dS_{q_1,q_2}(p) = 0 \end{array} \right. \tag{10-81}$$

dS is called an "abelian differential of the third kind".

Starting from the behaviours near poles and irreducible cycles 10-80, we easily find:

$$\Omega(p)dx(p) = \left.\frac{\partial \omega(p)dx(p)}{\partial T}\right|_{x(p)} = -dS_{\infty_+,\infty_-}(p) \tag{10-82}$$

$$H(p,p_\xi)dx(p) = \left.\frac{\partial \omega(p)dx(p)}{\partial h}\right|_{x(p)} = -dS_{\overline{p}_\xi,\infty_-}(p) = dS_{p_\xi,\infty_+}(p) - d\ln\left(x(p) - x(p_\xi)\right) \tag{10-83}$$

Theorem 10.2 *On an algebraic curve of genus g, there exist exactly g linearly independent "holomorphic differential forms" (i.e. with no poles), $du_i(p)$, $i = 1,\dots,g$. They can be chosen normalized as:*

$$\oint_{\mathcal{A}_i} du_j(p) = \delta_{ij} \tag{10-84}$$

For hyperelliptical surfaces, it is easy to see that if $L(x)$ is a polynomial of degree at most $g-1 = s-2$, the differential form $\frac{L(x)}{\sqrt{\prod_{i=1}^s (x-a_i)(x-b_i)}}dx$ is regular at ∞, at the branch points, and thus has no poles. And there are g linearly independent polynomials of degree at most $g-1$. The irreducible cycles $\mathcal{A}_i$ is a contour surrounding $[a_i, b_i]$ in the positive direction.

Definition 10.1 *The matrix of periods is defined by:*

$$\tau_{ij} := \oint_{\mathcal{B}_i} du_j(p) \tag{10-85}$$

where the irreducible cycles $\mathcal{B}_i$ are chosen canonicaly conjugated to the $\mathcal{A}_i$, i.e. $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij}$. In our hyperelliptical case, we choose $\mathcal{B}_i$ as a contour crossing $[a_i, b_i]$ and $[a_s, b_s]$.

The matrix of periods is symmetric $\tau_{ij} = \tau_{ji}$, and its imaginary part is positive $\Im\tau_{ij} > 0$. It encodes the complex structure of the curve.

The holomrphic forms naturaly define an embedding of the curve into $\mathbb{C}^g$:

Definition 10.2 *Given a base point $q_0 \in \mathcal{E}$, we define the Abel map:*

$$\begin{array}{rcl} \mathcal{E} & \longrightarrow & \mathbb{C}^g \\ p & \longrightarrow & \vec{u}(p) = (u_1(p), \ldots, u_g(p)) \quad , \quad u_i(p) := \int_{q_0}^{p} du_i(p) \end{array} \tag{10-86}$$

where the integration path is chosen so that it does not cross any $\mathcal{A}_i$ or $\mathcal{B}_i$.

Definition 10.3 *Given a symmetric matrix τ of dimension g, such that $\Im\tau_{ij} > 0$, we define the θ-function, from $\mathbb{C}^g \to \mathbb{C}$ by:*

$$\theta(\vec{u}, \tau) = \sum_{\vec{m}\in\mathbb{Z}^g} \mathbf{e}^{i\pi\vec{m}^t\tau\vec{m}}\, \mathbf{e}^{2i\pi\vec{m}^t\vec{u}} \tag{10-87}$$

It is an even entire function. For any $\vec{m} \in \mathbb{Z}^g$, it satisfies:

$$\theta(\vec{u}+\vec{m}) = \theta(\vec{u}) \quad , \quad \theta(\vec{u}+\tau\vec{m}) = \mathbf{e}^{-i\pi(2\vec{m}^t\vec{u}+\vec{m}^t\tau\vec{m})}\,\theta(\vec{u}) \tag{10-88}$$

Definition 10.4 *The theta function vanishes on a codimension 1 submanifold of $\mathbb{C}^g$, in particular, it vanishes at the odd half periods:*

$$\vec{z} = \frac{\vec{m}_1 + \tau\,\vec{m}_2}{2} \ , \ \vec{m}_1 \in \mathbb{Z}^g \ , \ \vec{m}_2 \in \mathbb{Z}^g \ , \ (\vec{m}_1^t\vec{m}_1) \in 2\mathbb{Z}+1 \quad \longrightarrow \quad \theta(\vec{z}) = 0 \tag{10-89}$$

For a given such odd half-period, we define the characteristic $\vec{z}$ θ-function:

$$\theta_{\vec{z}}(\vec{u}) := \mathbf{e}^{i\pi m_2\vec{u}+}\,\theta(\vec{u}+\vec{z}) \tag{10-90}$$

so that:

$$\theta_{\vec{z}}(\vec{u}+\vec{m}) = \mathbf{e}^{i\pi\vec{m}_2^t\vec{m}}\,\theta_{\vec{z}}(\vec{u}) \quad , \quad \theta_{\vec{z}}(\vec{u}+\tau\vec{m}) = \mathbf{e}^{-i\pi\vec{m}_1^t\vec{m}}\,\mathbf{e}^{-i\pi(2\vec{m}^t\vec{u}+\vec{m}^t\tau\vec{m})}\,\theta_{\vec{z}}(\vec{u}) \tag{10-91}$$

and

$$\theta_{\vec{z}}(\vec{0}) = 0 \tag{10-92}$$

Definition 10.5 *Given two points p, q in $\mathcal{E}$, as well as a basepoint $p_0 \in \mathcal{E}$ and an odd half period z, we define the prime form $E(p,q)$:*

$$E(p,q) := \frac{\theta_{\vec{z}}(\vec{u}(p)-\vec{u}(q))}{\sqrt{dh_{\vec{z}}(p)dh_{\vec{z}}(q)}} \tag{10-93}$$

where $dh_{\vec{z}}(p)$ is the holomorphic form:

$$dh_{\vec{z}}(p) := \sum_{i=1}^{g} \left.\frac{\partial\theta_{\vec{z}}(\vec{u})}{\partial u_i}\right|_{\vec{u}=\vec{0}} du_i(p) \tag{10-94}$$

Theorem 10.3 *The abelian differentials can be written:*

$$dS_{q_1,q_2}(p) = d\ln \frac{E(p,q_1)}{E(p,q_2)} \tag{10-95}$$

With these definitions, we have:

$$\Lambda(p) = \gamma\, \frac{\theta_{\vec{z}}(\vec{u}(p)-\vec{u}(\infty_-))}{\theta_{\vec{z}}(\vec{u}(p)-\vec{u}(\infty_+))} \quad , \quad \gamma := \lim_{p\to\infty_+} \frac{x(p)\,\theta_{\vec{z}}(\vec{u}(p)-\vec{u}(\infty_+))}{\theta_{\vec{z}}(\vec{u}(\infty_+)-\vec{u}(\infty_-))} \tag{10-96}$$

$$H(p_\xi) = \frac{\theta_{\vec{z}}(\vec{u}(p_\xi)-\vec{u}(\infty_-))\theta_{\vec{z}}(\vec{u}(\infty_+)-\vec{u}(\overline{p}_\xi))}{\theta_{\vec{z}}(\vec{u}(p_\xi)-\vec{u}(\overline{p}_\xi))\theta_{\vec{z}}(\vec{u}(\infty_+)-\vec{u}(\infty_-))} = -\gamma\, \frac{\theta_{\vec{z}}(\vec{u}(\infty_+)-\vec{u}(\infty_-))}{\theta_{\vec{z}}(\vec{u}(p_\xi)-\vec{u}(\infty_+))^2}\, \frac{dh_{\vec{z}}(p_\xi)}{dx(p_\xi)} \tag{10-97}$$

11 Asymptotics of orthogonal polynomials

11.1 One-cut case

In the one-cut case, (i.e. genus zero algebraic curve), and if V is a real potential, there is only one dominant saddle point if $\xi \notin [a,b]$, and two conjugated dominant saddle points if $x \in [a,b]$. More generaly, there is a saddle point corresponding to each determination of p_ξ such that $x(p_\xi) = \xi$. i.e. p_ξ and $\overline{p}_\xi = 1/p_\xi$. The dominant saddle point is the one such that $\Re(V_{\text{eff}}(p_\xi) - V(\xi))$ is minimal. The two cols have a contribution of the same order if:

$$\Re V_{\text{eff}}(p_\xi) = \Re V_{\text{eff}}(\overline{p}_\xi) \tag{11-98}$$

i.e. if ξ is such that:

$$\Re \int_{\overline{p}_\xi}^{p_\xi} W(x)dx = 0 \tag{11-99}$$

If the potential is real, it is easy to see that the set of points which satisfy 11-99 is $[a,b]$, in general, it is a curve in the complex plane, going from a to b, we call it the cut $[a,b]$ (similar curves were studied in [31]).

Then we have:

- For $x \notin [a,b]$, we write $\xi = \frac{a+b}{2} + \gamma(p_\xi + 1/p_\xi)$, $\gamma = \frac{b-a}{4}$:

  $$p_n(\xi) \sim \sqrt{H(p_\xi)}\, (\Lambda(p_\xi))^{n-N}\, \mathbf{e}^{-\frac{N}{2}(V_{\text{eff}}(p_\xi)-V(\xi))}(1+O(1/N)) \tag{11-100}$$

  i.e.

  $$p_n(\xi) \sim \sqrt{\frac{\gamma}{x'(p_\xi)}}\, (\gamma\, p_\xi)^{n-N}\, \mathbf{e}^{-\frac{N}{2}(V_{\text{eff}}(p_\xi)-V(\xi))}(1+O(1/N)) \tag{11-101}$$

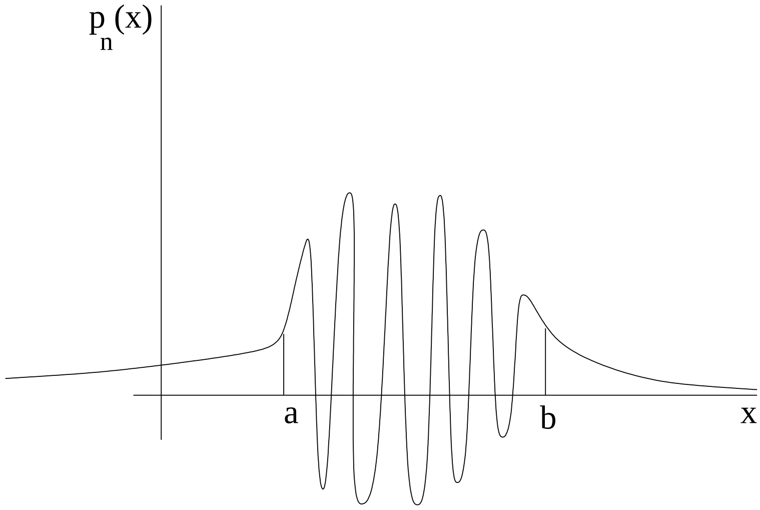
- For $x \in [a,b]$, i.e. p is on the unit circle $p = \mathbf{e}^{i\phi}$, $\xi = \frac{a+b}{2} + 2\gamma \cos\phi$:

$$\begin{aligned} p_n(\xi) \quad \sim \quad & \sqrt{H(p_\xi)}\,\left(\Lambda(p_\xi)\right)^{n-N}\,\mathbf{e}^{-\frac{N}{2}(V_{\rm eff}(p_\xi)-V(\xi))}(1+O(1/N)) \\ & + \sqrt{H(\overline{p}_\xi)}\,\left(\Lambda(\overline{p}_\xi)\right)^{n-N}\,\mathbf{e}^{-\frac{N}{2}(V_{\rm eff}(\overline{p}_\xi)-V(\xi))}(1+O(1/N)) \end{aligned} \tag{11-102}$$

i.e.

$$p_n(\xi) \sim \frac{\gamma^{n-N}}{\sqrt{2\sin\phi(\xi)}}\, 2\cos\left(N\pi\int_a^\xi \rho(x)dx - (n-N+\frac{1}{2})\phi(\xi)+\alpha\right)(1+O(1/N)) \tag{11-103}$$

i.e. we have an oscillatory behaviour



11.2 Multi-cut case

In the multicut case, in addition to having saddle-points corresponding to both determinantions of p_ξ, we have a saddle point for each filling fraction configuration $n_1, \ldots, n_s$ with $\sum_{i=1}^s n_i = n$. We write:

$$\epsilon_i = \frac{n_i}{N} \tag{11-104}$$

The saddle point corresponding to filling fractions which differ by a few units, contribute to the same order, and thus cannot be neglected. One has to consider the sommation over filling fractions [10].

Thus, one has to consider the action of a saddle point as a function of the filling fractions. We leave as an exercise for the reader to prove that the derivatives of F are given by:

$$\frac{\partial F}{\partial \epsilon_i} = -\oint_{\mathcal{B}_i} W(x)dx \tag{11-105}$$

and:

$$\frac{\partial^2 F}{\partial\epsilon_i \partial T} = -2i\pi(u_i(\infty_+) - u_i(\infty_-)) \tag{11-106}$$

$$\frac{\partial^2 F}{\partial\epsilon_i \partial h} = -2i\pi(u_i(p_\xi) - u_i(\infty_+)) \tag{11-107}$$

$$\frac{\partial^2 F}{\partial\epsilon_i \partial\epsilon_j} = -2i\pi\tau_{ij} \tag{11-108}$$

The last relation implies that $\Re F$ is a convex function of ϵ, thus it has a unique minimum:

$$\vec{\epsilon}^* \quad , \quad \Re \left.\frac{\partial F}{\partial\epsilon_i}\right|_{\vec{\epsilon}=\vec{\epsilon}^*} = 0 \tag{11-109}$$

We write:

$$\zeta_i := -\frac{1}{2i\pi} \left.\frac{\partial F}{\partial\epsilon_i}\right|_{\vec{\epsilon}=\vec{\epsilon}^*} \quad , \quad \zeta_i \in \mathbb{R} \tag{11-110}$$

We thus have the Taylor expansion:

$$\begin{aligned} F(T,h,\vec{\epsilon}) \quad \sim \quad & F(1,0,\vec{\epsilon}^*) - 2i\pi\vec{\zeta}^t(\vec{\epsilon}-\vec{\epsilon}^*) + (T-1)\frac{\partial F}{\partial T} + \frac{h}{2}(V_{\text{eff}}(p_\xi) - V(\xi)) \\ & + \frac{(T-1)^2}{2}\frac{\partial^2 F}{\partial T^2} - (T-1)h\ln\Lambda(p_\xi) - \frac{h^2}{2}\ln H(p_\xi) \\ & -2i\pi(\vec{\epsilon}-\vec{\epsilon}^*)^t\tau(\vec{\epsilon}-\vec{\epsilon}^*) - 2i\pi(T-1)(\vec{\epsilon}-\vec{\epsilon}^*)^t(\vec{u}(\infty_+) - \vec{u}(\infty_-)) \\ & -2i\pi h(\vec{\epsilon}-\vec{\epsilon}^*)^t(\vec{u}(p_\xi) - \vec{u}(\infty_+)) + \dots \end{aligned} \tag{11-111}$$

Thus:

$$\begin{aligned} Z \quad & \sim \quad \sum_I C_I \mathbf{e}^{-N^2 F(\{x\}_I)} \\ & \sim \quad \sum_{p=p_\xi,\overline{p}_\xi} \mathbf{e}^{-N^2F(1,0,\vec{\epsilon}^*)} \mathbf{e}^{N^2\left(-(T-1)\frac{\partial F}{\partial T} - \frac{h}{2}(V_{\text{eff}}(p)-V(\xi)) - \frac{(T-1)^2}{2}\frac{\partial^2 F}{\partial T^2} + (T-1)h\ln\Lambda(p) + \frac{h^2}{2}\ln H(p)\right)} \\ & \qquad \sum_{\vec{n}} \mathbf{e}^{i\pi(\vec{n}-N\vec{\epsilon}^*)^t\tau(\vec{n}-N\vec{\epsilon}^*)} \mathbf{e}^{2i\pi N\vec{\zeta}^t(\vec{n}-N\vec{\epsilon}^*)} \\ & \qquad \mathbf{e}^{2i\pi N(T-1)(\vec{n}-N\vec{\epsilon}^*)^t(\vec{u}(\infty_+)-\vec{u}(\infty_-))} \mathbf{e}^{2i\pi Nh(\vec{n}-N\vec{\epsilon}^*)^t(\vec{u}(p)-\vec{u}(\infty_+))} \end{aligned}$$

$(11-112)$

In that last sum, because of convexity, only values of $\vec{n}$ which don't differ from $N\vec{\epsilon}^*$ form more than a few units, contribute substantialy. Therefore, up to a non perturbative error (exponentialy small with N), one can extend the sum over the n_i's to the whole $\mathbb{Z}^g$, and recognize a θ-function (see 10-87):

$$Z \quad \sim \quad \sum_{p=p_\xi,\overline{p}_\xi} \mathbf{e}^{-N^2F(1,0,\vec{\epsilon}^*)} \mathbf{e}^{N^2\left((T-1)\frac{\partial F}{\partial T} + \frac{h}{2}(V_{\text{eff}}(p)-V(\xi)) + \frac{(T-1)^2}{2}\frac{\partial^2 F}{\partial T^2} + (T-1)h\ln\Lambda(p) + \frac{h^2}{2}\ln H(p)\right)}$$

$$\mathbf{e}^{i\pi N^2 \vec\epsilon^{*t}\tau\vec\epsilon^*}\,\mathbf{e}^{-2i\pi N^2\vec\zeta^t\vec\epsilon^*}\,\mathbf{e}^{-2i\pi N^2(T-1)\vec\epsilon^{*t}(\vec u(\infty_+)-\vec u(\infty_-))}\,\mathbf{e}^{-2i\pi N^2 h\vec\epsilon^{*t}(\vec u(p)-\vec u(\infty_+))}$$
$$\theta(N(\vec\zeta-\tau\vec\epsilon^*)+N(T-1)(\vec u(\infty_+)-\vec u(\infty_-))+Nh(\vec u(p)-\vec u(\infty_+)),\tau) \qquad (11-113)$$

with $T-1=\frac{n-N}{N}$ and $h=0$ or $h=1/N$, we get the asymptotics:

$$p_n(\xi) \sim \sum_{x(p)=\xi} \sqrt{H(p)}\,(\Lambda(p))^{n-N}\,\mathbf{e}^{-\frac{N}{2}(V_{\rm eff}(p)-V(\xi))}\,\mathbf{e}^{-2i\pi N\vec\epsilon^{*t}(\vec u(p)-\vec u(\infty_+))}$$
$$\frac{\theta(N(\vec\zeta-\tau\vec\epsilon^*)+(n-N)(\vec u(\infty_+)-\vec u(\infty_-))+(\vec u(p)-\vec u(\infty_+)),\tau)}{\theta(N(\vec\zeta-\tau\vec\epsilon^*)+(n-N)(\vec u(\infty_+)-\vec u(\infty_-)),\tau)} \qquad (11-114)$$

Again, depending on ξ, we have to choose the determination of p_ξ which has the minimum energy. If we are on a cut, i.e. if condition 11-99 holds, both determinations contribute. To summarize, outside the cuts, the sum 11-114 reduces to only one term, and along the cuts, the sum 11-114 contains two terms.

12 Conclusion

We have shown how the asymptotics of orthogonal polynomials (a notion related to integrability) is deeply related to algebraic geometry. This calculation can easily be extended to many generalizations, for multi-matrix models [17, 16, 2, 18], non-hermitean matrices ($\beta=1,4$) [20], rational potentials [3], ...

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